1 The Catchup Algorithm

The catchup algorithm is the method for slewing an axis (telescope or delay line) moving at velocity v_i at position 0 so that it intercepts a target which moves with constant velocity v_f and which has a head start of distance d_0, subject to the constraints of a maximum acceleration a_m and a maximum velocity v_m. a_m and v_m are positive numbers, v_i, v_f, and d_0 can be positive, negative, or zero. At the end of the slew, the axis should have both the same position and velocity as the target.

1.1 Will it overshoot?

It is possible that the axis will overshoot the target if it's initial velocity is too fast and the target's head start is too small. Recall from elementary one dimensional kinematics that a body which accelerates constantly from inital velocity v_i to final velocity v_f covers a distance

$$d = \frac{v_f^2 - v_i^2}{2a}. \tag{1}$$

Then the distance covered decelerating ($a = -a_m$) from v_i to v_f is

$$\frac{v_i^2 - v_f^2}{2a_m} \tag{2}$$

and the deceleration takes a time $(v_i - v_f)/a_m$ during which the target moves to position

$$d_0 + \frac{v_i - v_f}{a_m} v_f. \tag{3}$$

So clearly the condition for overshooting the target is

$$\frac{v_i^2 - v_f^2}{2a_m} > d_0 + \frac{v_i - v_f}{a_m} v_f \tag{4}$$

or

$$(v_i - v_f)^2 > 2a_m d_0 \tag{5}$$

1.2 No maximum velocity

First, consider the case when there is no maximum velocity, or equivalently, where the target is so close that there is not enough time to accelerate to maximum velocity. In this case the algorithm could be described as follows: accelerate constantly at $a = \pm a_m$ from time 0 until time t_0, reaching velocity $v_i + at_0$; then decelerate constantly at $-a$ until reaching velocity v_f at the target's position. The time required to decelerate is $[v_f - (v_i + at_0)]/-a$; the catchup is achieved at time

$$t_0 + \left[\frac{v_f - (v_i + at_0)}{-a}\right] = 2t_0 + \frac{v_i - v_f}{a} \tag{6}$$

at which time the target has moved to position

$$d_0 + \left(2t_0 + \frac{v_i - v_f}{a}\right) v_f \tag{7}$$

A distance

$$\frac{(v_i + at_0)^2 - v_i^2}{2a} \tag{8}$$

is covered ramping up and a distance

$$\frac{v_f^2 - (v_i + at_0)^2}{-2a} \tag{9}$$

ramping down. If the final position is to coincide with the target's then

$$\frac{(v_i + at_0)^2 - v_i^2}{2a} + \frac{v_f^2 - (v_i + at_0)^2}{-2a} = d_0 + \left(2t_0 + \frac{v_i - v_f}{a}\right) v_f \tag{10}$$

which is a quadratic equation for t_0. A little bit of algebra allows us to rearrange this equation as

$$at_0^2 + 2(v_i - v_f)t_0 + \frac{(v_i - v_f)^2}{2a} - d_0 = 0 \tag{11}$$

which has solutions

$$t_0 = \frac{(v_f - v_i)}{a} \pm \sqrt{\frac{(v_i - v_f)^2}{2a^2} + \frac{d_0}{a}}. \tag{12}$$

1.3 Maximum velocity v_m

If a maximum velocity is enforced, then the catchup algorithm is described as follows: accelerate constantly at a_m from time 0 until time $(v_m - v_i)/a_m$; then maintain a contant velocity v_m until time t_0; then decelerate constantly at a_m until reaching velocity v_f at the target's position. The time required to decelerate to v_f is $(v_m - v_f)/a_m$; the catchup is achieved at time

$$t_0 + \frac{v_m - v_f}{a_m} \tag{13}$$

at which time the target has moved to position

$$d_0 + \left(t_0 + \frac{v_m - v_f}{a_m}\right) v_f \tag{14}$$

A distance

$$\frac{v_m^2 - v_i^2}{2a_m} \tag{15}$$

is covered ramping up, a distance

$$\frac{v_m^2 - v_f^2}{2a_m} \tag{16}$$

ramping down, and a distance

$$\left(t_0 - \frac{v_m - v_i}{a_m}\right) v_m \tag{17}$$

at constant velocity. Since the final position coincides with the target position

$$\frac{v_m^2 - v_i^2}{2a_m} + \frac{v_m^2 - v_f^2}{2a_m} + \left(t_0 - \frac{v_m - v_i}{a_m}\right) v_m = d_0 + \left(t_0 + \frac{v_m - v_f}{a_m}\right) v_f \tag{18}$$

Solving this equation for t_0 gives

$$t_0 = \frac{1}{v_m - v_f}\left[d_0 + \frac{v_i^2 + 2v_m(v_f - v_i) - v_f^2}{2a_m}\right] \tag{19}$$